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A Necessary Condition for Best Approximation in Monotone and Sign-Monotone Norms

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Best approximation to $f \in C[a, b]$ by elements of an n-dimensional Tchebycheff space in monotone norms (norms defined on $C[a, b]$ for which $|f(x)| \leqslant |g(x)|$, $a \leqslant x \leqslant b$, implies $\|f\| \leqslant \|g\|$) is studied. It is proved that the error function has at least n zeroes in $[a, b]$, counting twice interior zeroes with no change of sign. This result is best possible for monotone norms in general, and improves the one in [5]. The proof follows from the observation that, for any monotone norm, $\operatorname{sgn} f(x) = \operatorname{sgn} g(x)$, $a \leqslant x \leqslant b$, implies $\|f - \lambda g\| < \|f\|$ for $\lambda > 0$ small enough. This property is shown to characterize a class of norms wider than the class of monotone norms, namely "sign-monotone" norms defined by: $|f(x)| \leqslant |g(x)|$, $f(x)\, g(x) \geqslant 0$, $a \leqslant x \leqslant b$, implies $\|f\| \leqslant \|g\|$. It is noted that various results concerning approximation in monotone norms, are actually valid for approximation in sign-monotone norms.

1. INTRODUCTION

In studying some problems of approximation in the L_p-norms, $1 \leqslant p \leqslant \infty$, we found that the same methods of proof are in fact applicable for approximation in a wider class of norms, namely, "monotone norms" [4, 5]. As introduced in [4], a monotone norm defined on $C[a, b]$ is a norm $\|\cdot\|$ for which

$$|f(x)| \leqslant |g(x)|, \qquad a \leqslant x \leqslant b, \qquad \text{implies} \quad \|f\| \leqslant \|g\|. \tag{1.1}$$

$$\| \, |f| \, \| = \|f\|, \tag{1.2}$$

$$\|f\| \leqslant \|f\|_\infty \| 1 \|, \tag{1.3}$$

and

$$\|f\| < \|g\| \quad \text{if} \quad |f(x)| < |g(x)|, \qquad a \leqslant x \leqslant b. \tag{1.4}$$

In [5] monotone norms are found to be closely related to the concept of "Fejér monotonic norm" [2]. More specifically, it is proved there that

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a norm $\|\cdot\|$ is monotone if and only if it has *the Fejér property*, namely: $|f(x)| \leqslant |g(x)|$, $a \leqslant x \leqslant b$, with equaltiy only at the zeroes of $g(x)$, implies $\|f\| < \|g\|$ if $g \not\equiv 0$. By means of this result the following necessary condition for best approximation in monotone norms from

$$P = \{p \mid p \in \Pi_{n-1}, l(x) \leqslant p(x) \leqslant u(x), a \leqslant x \leqslant b\} \tag{1.5}$$

is derived (Π_{n-1} is the class of all polynomials of degree $\leqslant n-1$):

Result A [5]. Let $f \in C^{n-1}[a, b]$ satisfy $l(x) \leqslant f(x) \leqslant u(x)$, $a \leqslant x \leqslant b$, and let p^* be a polynomial of best approximation to f from P in a monotone norm. Then if $f - p^*$ has only isolated zeroes,

$$z^*(f - p^*) \geqslant n. \tag{1.6}$$

($z^*(g)$ denotes the number of isolated zeroes in $[a, b]$ of $g \in C^{n-1}[a, b]$, counting multiplicites up to order n.)

In the special case of approximation in L_p-norms, $1 \leqslant p \leqslant \infty$, without restrictions on the range, the specific characterizations of best approximations yield the stronger result that the number of changes of sign of $f - p^*$ is at least n. The following example indicates that this is not the case for monotone norms in general.

EXAMPLE [5]. Let $\|\cdot\|$ be a monotone norm defined on $C[-1, 1]$ by

$$\|f\| = \sup_{-1 \leqslant x \leqslant 1} |f(x)| + 2\,|f(0)|,$$

and let $f(x) = x^2$; $-1 \leqslant x \leqslant 1$. It is easily seen that $p^* \equiv 0$ is the unique polynomial of best approximation to f from Π_1. Thus $z^*(f - p^*) = 2$, while $f - p^*$ has no changes of sign.

Although in this example (1.6) holds with equality, yet Result A is not best possible. The main purpose of this work is to show that for functions lying strictly inside the range, $z^*(f - p^*)$ in (1.6) may be replaced by $\tilde{z}(f - p^*)$ the number of isolated zeroes of $f - p^*$ in $[a, b]$, counting twice zeroes in (a, b) at which $f - p^*$ does not change sign. Moreover, the improved result is not limited to differentiable functions and is proved in the more general context of approximation by elements from the span of a Tchbysheff-system (T-system) $\{v_0, v_1, ..., v_{n-1}\}$ with one of the following properties:

$$\{v_0, v_1, ..., v_{n-2}\} \text{ is also a } T\text{-system on } [a, b], \tag{1.7}$$

$$\{v_0, v_1, ..., v_{n-1}\} \text{ is a } T\text{-system on } [a', b'], \qquad a' < a < b < b', \tag{1.8}$$

or

$$\{v_0, v_1, ..., v_{n-1}\} \text{ is extended of order 2 on } [a, b]. \tag{1.9}$$

(For the definition and basic properties of T-systems the reader is referred to [3, Chap. 1]).

In Section 2, the main result of this work is proved by showing that every monotone norm has the following property:

$$\operatorname{sgn} f(x) = \operatorname{sgn} g(x), \quad a \leqslant x \leqslant b, \quad \text{implies } \|f - \lambda g\| < \|f\| \quad \text{for} \quad \lambda > 0 \text{ small enough.} \tag{1.10}$$

It is observed that this property characterizes a class of norms wider than the class of monotone norms. It is shown that these norms, termed "sign-monotone," can also be defined by:

$$f(x) \cdot g(x) \geqslant 0, \quad |f(x)| \leqslant |g(x)|, \quad a \leqslant x \leqslant b, \qquad \text{imply} \quad \|f\| \leqslant \|g\|. \tag{1.11}$$

This leads to the conclusion that the main result of Section 2 which depends on property (1.10) as well as other results depending on property (1.11) of the norm are valid in the general setting of approximation in sign-monotone norms.

2. Main Results

First we show that all monotone norms share a property which is self-evident in case of the sup-norm, but not obvious even for the L_p-norms, $1 \leqslant p \leqslant \infty$.

Theorem 2.1. *Let $f, g \in C[a, b]$ satisfy $f, g \not\equiv 0$,*

$$\operatorname{sgn}[f(x)] = \operatorname{sgn}[g(x)], \qquad a \leqslant x \leqslant b, \qquad (\operatorname{sgn}(0) = 0), \tag{2.1}$$

and let $\|\cdot\|$ be a given monotone norm. Then, for every $\lambda > 0$ small enough

$$\|f - \lambda g\| < \|f\|. \tag{2.2}$$

Proof. In case f and g do not vanish on $[a, b]$, then for any $0 < \lambda < m = \min_{a \leqslant x \leqslant b} (f(x)/g(x))$

$$|f(x) - \lambda g(x)| < |f(x)|, \qquad a \leqslant x \leqslant b, \tag{2.3}$$

and the claim of the theorem follows from (1.4). If f and g have zeros in $[a, b]$, a more refined analysis is required.

Let us consider the set $A = \{x \mid |f(x)| < \|f\|/(4\|1\|)\}$.

Obviously A is an open set containing all the zeros of f (and g), and in view of (1.3) $A \neq (a, b)$. Hence, $B = [a, b] - A$ is a closed non-empty proper subset of $[a, b]$, on which $g(x)/f(x) \neq 0$, and by (2.1) $\min_{x \in B} g(x)/f(x) = \bar{m} > 0$.

For a fixed value of μ, $0 < \mu < \min(1, \bar{m}\}$, define the set

$$H_1 = \{x \mid \mu \mid f(x) \mid \geqslant \mid g(x) \mid\}.$$

By the choice of μ, $H_1 \subset A$ and therefore

$$\max_{x \in H_1} \mid f(x) \mid \leqslant \frac{\| f \|}{4 \| 1 \|}. \tag{2.4}$$

Using similar arguments, we can find $0 < \lambda^* < 1$ such that $\mid g(x) \mid \leqslant \mu \| f \| / (4 \| 1 \|)$ on the set $H_2(\lambda^*)$, where

$$H_2(\lambda) = \{x \mid \lambda \mid g(x) \mid \geqslant \mid f(x) \mid\}. \tag{2.5}$$

Obviously for any $\lambda < \lambda^*$, $H_2(\lambda) \subset H_2(\lambda^*)$, and therefore

$$\max_{x \in H_2(\lambda)} \mid g(x) \mid \leqslant \frac{\mu \| f \|}{4 \| 1 \|}, \qquad 0 < \lambda \leqslant \lambda^*. \tag{2.6}$$

Now, let $H_2 \equiv H_2(\lambda)$ for a fixed λ, $0 < \lambda \leqslant \lambda^*$.

By the construction of H_1 and H_2

$$\mid f(x) \mid \geqslant \lambda\mu \mid f(x) \mid \geqslant \lambda \mid g(x) \mid, \qquad \text{for} \quad x \in H_1, \tag{2.7}$$

$$\lambda \mid g(x) \mid \geqslant \mid f(x) \mid \geqslant \lambda\mu \mid f(x) \mid, \qquad \text{for} \quad x \in H_2, \tag{2.8}$$

and

$$\mid f(x) \mid > \lambda \mid g(x) \mid > \lambda\mu \mid f(x) \mid, \qquad \text{for} \quad x \in [a, b] - (H_1 \cup H_2). \tag{2.9}$$

(Note that $H_1 \cap H_2$ coincides with the set of zeros of f and g).

To see that (2.2) is valid we estimate $f - \lambda g$ differently on H_1, H_2 and $[a, b] - (H_1 \cup H_2)$, by decomposing it into the sum $h_1 + h_2$ where:

$$h_1(x) = \begin{cases} f(x) - \lambda g(x), & x \in [a, b] - (H_1 \cup H_2), \\ 0, & x \in H_2, \\ f(x) - \lambda\mu f(x), & x \in H_1, \end{cases} \tag{2.10}$$

$$h_2(x) = \begin{cases} 0, & x \in [a, b] - (H_1 \cup H_2), \\ f(x) - \lambda g(x), & x \in H_2, \\ \lambda\mu f(x) - \lambda g(x), & x \in H_1, \end{cases} \tag{2.11}$$

By (2.7), (2.8) and (2.9) $h_1, h_2 \in C[a, b]$. From (2.1), (2.9) and (2.10)

$$\mid h_1(x) \mid \leqslant \mid f(x) - \lambda\mu f(x) \mid, \qquad a \leqslant x \leqslant b,$$

and therefore

$$\| h_1 \| \leqslant \|f - \lambda\mu f\| = (1 - \lambda\mu) \|f\|. \tag{2.12}$$

On the other hand, by (2.1) (2.7) and (2.8)

$$| h_2(x)| \leqslant \begin{cases} \lambda\mu \ |f(x)|, & x \in H_1, \\ \lambda \ | g(x)|, & x \in H_2, \end{cases}$$

and in view of (2.4), (2.6) and (2.11)

$$| h_2(x)| \leqslant \frac{\lambda\mu \|f\|}{4 \| 1 \|}, \qquad a \leqslant x \leqslant b.$$

Therefore by (1.3)

$$\| h_2 \| \leqslant \tfrac{1}{4}\lambda\mu \|f\| \tag{2.13}$$

which together with (2.12) lead to

$$\|f - \lambda g \| \leqslant \| h_1 \| + \| h_2 \| \leqslant (1 - \lambda\mu) \|f\| + \tfrac{1}{4}\lambda\mu \|f\| < \|f\|.$$

The above elaborate proof is similar but more refined than the one used in [5] to show that every monotone norm satisfies the Fejér property. In fact this can be shown easily by means of the last theorem.

Using Theorem 2.1, we are able to improve Result A only for functions lying strictly inside the range.

THEOREM 2.2. *Let $f \in C[a, b]$ satisfy*

$$l(x) < f(x) < u(x), \qquad a \leqslant x \leqslant b, \tag{2.14}$$

and assume

$$\eta = \min\{ \inf_{a \leqslant x \leqslant b} \{u(x) - f(x)\}, \inf_{a \leqslant x \leqslant b} \{f(x) - \ell(x)\}\} > 0.$$

Let $\{v_0, v_1, ..., v_{n-1}\}$ be a T-system with one of the properties (1.7), (1.8) *or* (1.9) *and let*

$$V = \left\{ v \mid v(x) = \sum_{i=0}^{n-1} c_i v_i(x), \ell(x) \leqslant v(x) \leqslant u(x), a \leqslant x \leqslant b \right\}.$$

If v^* is an element of best approximation to f from V in a monotone norm $\| \cdot \|$, and $f - v^*$ has only isolated zeros, then

$$\tilde{z}(f - v^*) \geqslant n. \tag{2.15}$$

Proof. Assume $\tilde{z}(f - v^*) < n$ and let $z_1, z_2, ..., z_l$, $l < n$, be the zeros of $f - v^*$. By [3 Chap. 1, Theorem 5.2] and one of the assumptions (1.7), (1.8) or (1.9), it is possible to construct $q(x) = \sum_{i=0}^{n-1} a_i v_i(x)$ satisfying $\operatorname{sgn} q(x) = \operatorname{sgn}(f(x) - v^*(x))$, $a \leqslant x \leqslant b$. Thus, by Theorem 2.1 there exists $0 < \lambda^* < 1$ such that for all λ, $0 < \lambda \leqslant \lambda^*$

$$\| f - v^* - \lambda q \| < \| f - v^* \|.$$

This will lead to a contradiction provided we show that

$$l(x) \leqslant v^*(x) + \lambda q(x) \leqslant u(x), \qquad a \leqslant x \leqslant b, \tag{2.16}$$

for all $\lambda > 0$ small enough.

To this end define

$$Q^+ = \{x \mid a \leqslant x \leqslant b, q(x) \geqslant 0\}, Q = \{x \mid a \leqslant x \leqslant b, q(x) \leqslant 0\}.$$

By the construction of $q(x)$:

$$l(x) \leqslant v^*(x) \leqslant f(x) < u(x) \qquad \text{for} \quad x \in Q^+$$

and

$$l(x) < f(x) \leqslant v^*(x) \leqslant u(x) \qquad \text{for} \quad x \in Q^-.$$

Therefore, choosing $0 < \lambda < \min\{\lambda^*, \eta/\| q \|_\infty\}$ we have for $x \in Q^+$:

$$l(x) \leqslant v^*(x) \leqslant v^*(x) + \lambda q(x) \leqslant f(x) + \lambda \| q \|_\infty \leqslant u(x)$$

while for $x \in Q^-$

$$l(x) \leqslant f(x) - \lambda \| q \|_\infty \leqslant v^*(x) + \lambda q(x) \leqslant v^*(x) \leqslant u(x).$$

This completes the proof of the theorem.

From this proof it is clear that Theorem 2.2 is valid for approximation in any norm $\| \cdot \|$ for which $\operatorname{sgn} f(x) = \operatorname{sgn} g(x)$, $f \not\equiv 0$, $a \leqslant x \leqslant b$, implies $\| f - \lambda g \| < \| f \|$ for $\lambda > 0$ small enough. This property is typical not only to monotone norms but characterizes a wider class of norms. In fact, arguments similar to those employed in the proof of Theorem 2.1, lead to

THEOREM 2.3. *A norm $\| \cdot \|$ defined on $C[a, b]$ has the property*

$$\operatorname{sgn} f(x) = \operatorname{sgn} g(x), \quad a \leqslant x \leqslant b, \quad f \not\equiv 0, \quad \text{implies } \| f - \lambda g \| < \| f \| \quad \text{for } \lambda > 0 \text{ small enough} \tag{2.17}$$

if and only if it satisfies

$$f(x)\, g(x) \geqslant 0, \quad | f(x)| \leqslant | g(x)|, \quad a \leqslant x \leqslant b, \quad \text{implies } \| f \| \leqslant \| g \|. \tag{2.18}$$

Norms with property (2.18) are termed hereafter "sign-monotone norms". It follows directly from the definition that a norm is monotone if and only if it is sign-monotone and "absolute", where the term "absolute" refers to the property $\| \, |f| \, \| = \|f\|$. (See also [6, Chap. 6]). We conclude by showing that the class of sign-monotone norms is indeed wider than the class of monotone norms. Consider norms of the norm

$$N(f) = \|f^+\| + \|f^-\|,$$

where $\| \cdot \|$ is any monotone norm and $f^{\pm} = (f \pm |f|)/2$. Obviously $N(f)$ is a sign-monotone norm but is not absolute and therefore not monotone.

It follows from Theorem 2.3 that Theorem 2.2 and other results which rely on property (2.17) of monotone norms, (e.g. in [4, 5] and [1, p. 40]) are in fact valid in the more general context of sign-monotone norms.

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